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Safety-Critical Control as a Design Paradigm for Anytime Solvers of Variational Inequalities

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Abstract— **This paper shows that safety-critical control methods for the synthesis of safeguarding controllers also have an important role to play in optimization theory. We are motivated by applications where the solution of a variational inequality is used to regulate a dynamically evolving plant. We design algorithms to solve these problems by blending techniques from safety-critical control with tools from monotone operator theory. These algorithms are anytime, meaning that, if initialized in the feasible set, they are guaranteed to return a feasible solution to the optimization problem regardless of when they are terminated, which makes them particularly suitable for real-time applications. In some cases, our approach leads to reinterpretations of well-known algorithms from the lens of control theory, and in other cases we derive entirely novel algorithms. Our results demonstrate the promising potential of safety-critical control for both the analysis and design of optimization algorithms.**

I. Introduction

Convex optimization is of fundamental importance in engineering and applied science, and developing and analyzing numerical methods to solve convex programs has been the subject of intense research over the last two decades. A particularly successful framework has been to view iterative algorithms as dynamical systems, and then use tools from systems and control theory to analyze them. We are particularly motivated by settings where the optimization problem is used to regulate a dynamically evolving plant [1] (e.g., providing setpoints, specifying optimization-based controllers, steering plant toward an optimal steady-state). This type of problem arises in application areas such as power systems [2], network congestion control [3], and traffic networks [4]. In these settings, the system-theoretic point of view of optimization algorithms is critical for establishing stability of the overall interconnected system.

We tackle here the problem of systematically deriving algorithms to solve monotone variational inequalities. Monotone variational inequalities generalize many important problems in convex optimization, including minimization of a convex function subject to constraints, characterizing the Nash equilibria of a game, and finding saddle points of convex-concave functions. Our technical results are developed with a view towards feedback optimization problems. Often, these types of problems incorporate constraints, which when violated would threaten the safe operation of the physical system. Thus for a real-time implementation, it is crucial that the algorithm is *anytime*, meaning that, if initialized in the feasible set, it is guaranteed to return a feasible point even if terminated prematurely. We show that the design of anytime algorithms to solve variational inequalities can equivalently be cast as a feedback control problem, and solved by applying tools from safety-critical control.

Related Work: Classical references on the dynamical systems perspective of optimization algorithms include [5], [6], [7]. The methods discussed in this paper are most closely related to differential inclusions involving monotone set-valued maps, originally introduced in [8]. These systems have been equivalently described as a projected dynamical systems [9] and complementarity systems [10], [11].

We employ techniques from safety-critical control, which refers to the problem of designing a feedback controller to ensure that the state of a system satisfies certain constraints. This problem has gotten considerable attention in recent years due to the enormous number of applications in areas such as robotics and automotive systems. The work [12] reviews set invariance in control. Control barrier functions identify the range of inputs that keep the state safe leading to the synthesis of feedback controller that enforce forward invariance and asymptotic stability of the safe set. A popular technique for synthesizing controllers which ensure safety uses the concept of control barrier functions, see [13], [14], [15] and references therein.

Statement of Contributions: We [1] show how tools from safety-critical control can be used to design algorithms, in the form of continuous-time dynamical systems, which solve monotone variational inequalities. The algorithms are *anytime*, meaning they maintain feasibility of the state at all times and can therefore be terminated at any time. This makes them well suited for real-time applications. The basic idea is to view the constraint set for the problem as a safety set for a control system, and then to design a feedback

[1]Throughout the paper we use the following notation. Let $\mathbb{R}$ denote the set of real numbers. For $v, w \in \mathbb{R}^n$, $v \leq w$ (resp. $v < w$) denotes $v_i \leq w_i$ (resp. $v_i < w_i$) for $i \in \{1, \dots, n\}$. We let $\|v\|$ denote the Euclidean norm. We write $A \succeq 0$ (resp., $A \succ 0$) to denote A is positive semidefinite (resp., A is positive definite). For a symmetric matrix Q, $\lambda_{\min}(Q)$ denotes the minimum eigenvalue of Q. Given $g : \mathbb{R}^n \to \mathbb{R}$, we denote its gradient by ∇g. For $g : \mathbb{R}^n \to \mathbb{R}^m$, $\frac{\partial g(x)}{\partial x}$ denotes its Jacobian. For $I \subset \{1, 2, \dots, m\}$, we denote by $\frac{\partial g_I(x)}{\partial x}$ the matrix whose rows are $\{\nabla g_i(x)^\top\}_{i \in I}$. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ and a function $V : \mathbb{R}^n \to \mathbb{R}$, the *Lie derivative* of V along F is $\mathcal{L}_F V(x) = \nabla V(x)^\top F(x)$. Given a subset $C \subset \mathbb{R}^n$, the distance of $x \in \mathbb{R}^n$ to C is $\text{dist}(x, C) = \inf_{y \in C} \|x - y\|$. We let $\overline{C}$, $\text{int}(C)$, and ∂C denote the closure, interior, and boundary of C, respectively. The *projection map onto* $\overline{\mathcal{C}}$ is $\Pi_{\mathcal{C}} : \mathbb{R}^n \rightrightarrows \overline{\mathcal{C}}$, where $\Pi_{\mathcal{C}}(x) = \{y \in \overline{\mathcal{C}} \mid \|x - y\| = \text{dist}(x, \mathcal{C})\}$. Let $\mathcal{C} \subset \mathbb{R}^n$ be a closed and convex set. The *normal cone* to $\mathcal{C}$ at $x \in \mathbb{R}^n$ is $N_{\mathcal{C}}(x) = \{d \in \mathbb{R}^n \mid d^\top (x' - x) \leq 0, \ \forall x' \in \mathcal{C}\}$, and the *tangent cone* to $\mathcal{C}$ at x is $T_{\mathcal{C}}(x) = \{\xi \in \mathbb{R}^n \mid d^\top \xi \leq 0, \ \forall d \in N_{\mathcal{C}}(x)\}$.

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controller which maintains forward invariance of the set. We consider two approaches for designing such controllers. The first approach leads to a reinterpretation of projected methods from a control-theoretic lens, and the second approach leads to a novel class of algorithms. For both systems, we are able to establish a correspondence between equilibria and solutions of the variational inequality, global Lyapunov stability of the equilibria, and global asymptotic stability in the case of strong monotinicity. We demonstrate our methods on a convex optimization problem. For reasons of space, the proofs of the main results are omitted and will appear elsewhere.

II. Preliminaries

Here, we present basic notions on invariance, stability, and monotone variational inequalities.

A. Invariance and Stability Notions

We recall basic definitions from the theory of ordinary differential equations [16]. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz vector field and consider the dynamical system $\dot{x} = G(x)$. Local Lipschitz continuity ensures that, for every initial condition $x_0 \in \mathbb{R}^n$, there exists $T > 0$ and a unique trajectory $x : [0, T] \to \mathbb{R}^n$ such that $x(0) = x_0$ and $\dot{x}(t) = G(x(t))$. If the solution exists for all $t \geq 0$, the solution is *complete*. In this case, the *flow map* is defined by $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi_t(x) = x(t)$, where $x(t)$ is the unique solution with $x(0) = x$.

A set $\mathcal{C} \subset \mathbb{R}^n$ is *forward invariant* if $x \in \mathcal{C}$ implies that $\Phi_t(x) \in \mathcal{C}$ for all $t \geq 0$. If $\mathcal{C}$ is forward invariant and $x^* \in \mathcal{C}$ is an equilibrium, x^* is *Lyapunov stable relative to* $\mathcal{C}$ if for every open set U containing x^*, there exists an open set $\bar{U}$ also containing x^* such that for all $x \in \bar{U} \cap \mathcal{C}$, $\Phi_t(x) \in U \cap \mathcal{C}$ for all $t > 0$. The equilibrium x^* is *asymptotically stable relative to* $\mathcal{C}$ if it is Lyapunov stable relative to $\mathcal{C}$ and there is an open set U containing x^* such that $\Phi_t(x) \to x^*$ as $t \to \infty$ for all $x \in U \cap \mathcal{C}$. For all the concepts introduced here, when the invariant set is unspecified, we mean $\mathcal{C} = \mathbb{R}^n$. Analogous definitions of Lyapunov stability and asymptotically stability can be made for sets, instead of individual points.

B. Monotone Variational Inequalities

Here we review the basic theory of variational inequalities and monotone maps [17]. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map and $\mathcal{C} \subset \mathbb{R}^n$ a set playing the role of constraint set. A *variational inequality* is the problem of finding $z^* \in \mathcal{C}$ such that

$$(z - z^*)^\top F(z^*) \geq 0 \qquad \forall z \in C. \tag{1}$$

We denote this problem $\text{VI}(F, \mathcal{C})$ and define by $\text{SOL}(F, \mathcal{C})$ its set of solutions. We assume throughout that $\mathcal{C}$ is nonempty, closed and convex, in which case $z^* \in \text{SOL}(F, \mathcal{C})$ if and only if $0 \in F(z^*) + N_{\mathcal{C}}(z^*)$. One important example of variational inequality is the constrained optimization problem $\min_{x \in \mathcal{C}} f(x)$, which is equivalent to $\text{VI}(\nabla f, \mathcal{C})$.

We next provide a characterization of the solution set $\text{SOL}(F, \mathcal{C})$ for the special case where $\mathcal{C}$ is parameterized by inequality and affine equality constraints,

$$\mathcal{C} = \{z \in \mathbb{R}^n \mid g(z) \leq 0, h(z) = Az - b = 0\}, \tag{2}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^k$, $A \in \mathbb{R}^{n \times k}$, and $b \in \mathbb{R}^k$.Let $I(z) = \{1 \leq i \leq m \mid g_i(z) = 0\}$ denote the set of *active constraints at* $z \in \mathcal{C}$. The set $\mathcal{C}$ satisfies the *Mangasarian-Fromovitz Constraint Qualification* (MFCQ) condition at z, if $\{\nabla h_j(z)\}_{j=1}^k$ are linearly independent and there exists $\xi \in \mathbb{R}^n$ such that $\nabla h_j(z)^\top \xi = 0$ for all $j \in \{1, \ldots, k\}$ and $\nabla g_i(z)^\top \xi < 0$ for all $i \in I(z)$.

If MFCQ holds at z^*, then $z^* \in \text{SOL}(F, \mathcal{C})$ if and only if there exists $u^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^k$ such that

$$F(z^*) + \sum_{i=1}^m u_i^* \nabla g_i(z^*) + \sum_{i=1}^k v_i^* \nabla h_i(z^*) = 0 \tag{3a}$$

$$g(z^*) \leq 0 \tag{3b}$$

$$h(z^*) = 0 \tag{3c}$$

$$u^* \geq 0 \tag{3d}$$

$$(u^*)^\top g(z^*) = 0. \tag{3e}$$

These are the Karash-Kuhn-Tucker (KKT) conditions, and a triple (z^*, u^*, v^*) satisfying (3) is a *KKT triple*. The pair (u^*, v^*) are the *Lagrange multipliers* corresponding to z^*.

We are particularly interested in variational inequalities involving monotone maps. The map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is strongly monotone if

$$(z_1 - z_2)^\top (F(z_1) - F(z_2)) \geq 0,$$

for all $z_1, z_2 \in \mathbb{R}^n$, and F is *μ-strongly monotone* if there exists $\mu > 0$ such that

$$(z_1 - z_2)^\top (F(z_1) - F(z_2)) \geq \mu \left\| z_1 - z_2 \right\|^2,$$

for all $z_1, z_2 \in \mathbb{R}^n$.

When F is monotone, $\text{VI}(F, \mathcal{C})$ is called a *monotone variational inequality*. In this case $\text{SOL}(F, \mathcal{C})$ is convex if it as nonempty and when F is μ-strongly monotone, then $\text{SOL}(F, \mathcal{C})$ is at most a singleton.

III. Problem Statement

Our goal is to design algorithms that solve variational inequalities. We are motivated by real-time feedback optimization problems arising in application areas such as power systems, communication systems, and traffic networks, where the solution to the variational inequality is used to regulate a dynamically evolving process. However, for space reasons, we restrict our attention here to developing the theory in the context of an abstract variational inequality.

We want to design an algorithm in the form of continuous-time dynamical system $\dot{z} = G(z)$, such that trajectories of the system converge to solutions of the variational inequality (1). We want the algorithm to be *anytime*, meaning that, if initialized at a feasible point, it is guaranteed to return a feasible result, even if it is terminated before it converges to a solution. This is particularly important for real-time applications, where the result of the problem is used to regulate a physical plant, and constraints ensure its safe operation. Formally, the anytime property translates to the requirement that the feasible set $\mathcal{C}$ is forward invariant with respect to the dynamics. Finally, we seek to obtain stability guarantees for solutions, $z^* \in \text{SOL}(F, \mathcal{C})$, for the dynamics in question.

IV. SAFETY-CRITICAL CONTROL

We introduce here basic concepts from safety-critical control and introduce several methods for synthesizing safe-guarding feedback controllers. We build on these later to synthesize algorithms that solve variational inequalities. Consider a control-affine system

$$\dot{z} = \mathcal{F}(z, u) = F(z) + \sum_{i=1}^{r} u_i F_i(z), \tag{4}$$

with Lipschitz vector fields $F_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i \in \{0, \dots, r\}$, and a set $\mathcal{U} \subset \mathbb{R}^m$ of valid control inputs. Let $\mathcal{C} \subset \mathbb{R}^n$ represent the set of states where the system can operate safely and $u : \mathcal{Z} \to \mathcal{U}$ be a Lipschitz feedback controller, with $\mathcal{Z} \subset \mathbb{R}^n$ a set containing $\mathcal{C}$. The closed-loop system under u is *safe* with respect to $\mathcal{C}$ if $\mathcal{C}$ is forward invariant. In this case, we refer to u as a *safe-guarding* controller. Here we consider the problem of designing a safe-guarding feedback controller for (4) and provide two solutions to it.

A. Safeguarding Controller via Projection Onto Tangent Cone

The first strategy for synthesizing feedback controllers is via projection onto the tangent cone of $\mathcal{C}$, exploiting necessary and sufficient conditions for set-invariance for continuous-time dynamical systems [12].

Lemma 4.1 (Nagumo's Theorem): Let $\dot{z} = G(z)$ be a dynamical system and $\mathcal{C} \subset \mathbb{R}^n$ a closed convex set. Suppose that for each initial condition, the system admits a unique solution. Then, $\mathcal{C}$ is forward invariant under G if and only if $G(z) \in T_{\mathcal{C}}(z)$, for all $z \in \mathcal{C}$.

To design a feedback controller using Nagumo's Theorem, we define the map $K_{\text{proj}} : \mathbb{R}^n \rightrightarrows \mathcal{U}$ as

$$K_{\text{proj}}(z) := \Big\{ u \in \mathcal{U} \mid F(z) + \sum_{i=1}^{r} u_i F_i(z) \in T_{\mathcal{C}}(z) \Big\}.$$

The intuition is that any feedback $u : \mathcal{C} \to \mathcal{U}$ such that $u(z) \in K_{\text{proj}}(z)$ for $z \in \mathcal{C}$ will render $\mathcal{C}$ forward invariant. This is formalized next.

Lemma 4.2: (Projection-based Safeguarding Feedback): Consider the system (4) with safety set $\mathcal{C}$, and suppose that $K_{\text{proj}}(z) \neq \emptyset$ for all $z \in \mathcal{C}$. Then the feedback controller $u : \mathcal{C} \to \mathcal{U}$ is safeguarding if $u(z) \in K_{\text{proj}}(z)$ for all $z \in \mathcal{C}$, and the closed-loop system $\dot{z} = \mathcal{F}(z, u(z))$ admits a unique solution for all initial conditions.

While Lemma 4.2 provides sufficient conditions for a feedback controller to be safe, it does not specify how to synthesize it. For the special case where $\mathcal{C}$ can be parameterized in terms of inequality and equality constraints as in (2), we propose a strategy where $u(z)$ is expressed as the solution to a mathematical program. If MFCQ holds at $x \in \mathcal{C}$, the tangent cone can conveniently be expressed as $T_{\mathcal{C}}(z) = \{\xi \in \mathbb{R}^n \,\big|\, \frac{\partial h(z)}{\partial z}\xi = 0, \frac{\partial g_I(z)}{\partial z}\xi \leq 0\}$, in which case

$$K_{\text{proj}}(z) = \Big\{ u \in \mathcal{U} \mid \mathcal{L}_F g_i(z) + \sum_{\ell=1}^{r} u_\ell \mathcal{L}_{F_\ell} g_i(x) \leq 0,$$
$$\mathcal{L}_F h_j(z) + \sum_{\ell=1}^{r} u_\ell \mathcal{L}_{F_\ell} h_j(z) = 0, i \in I(z), 1 \leq j \leq k \Big\}.$$

Note that $K_{\text{proj}}(z)$ is defined in terms of affine constraints on the control input u. This suggests an optimization-based synthesis of a feedback satisfying the hypotheses of Lemma 4.2 by letting

$$u(z) \in \underset{u \in K_{\text{proj}}(z)}{\operatorname{argmin}} \Big\{ J(z, u) \Big\}, \tag{5}$$

for an appropriate choice of cost function $J : \mathcal{C} \times \mathcal{U} \to \mathbb{R}$. In general, care must be taken to ensure that the set K_{proj} is nonempty, and that $u(z)$ satisfies the appropriate regularity conditions to ensure existence and uniqueness for solutions of the resulting closed-loop dynamics. Additional complications arise when solutions to (5) are not unique. We address each of these challenges in the following sections.

B. Safeguarding Controller via Control Barrier Functions

The second strategy for synthesizing safeguarding feedback controllers uses the notion of vector control barrier functions. Let $\mathcal{C}$ be a safety set defined as in (2), and $\mathcal{C} \subset \mathcal{Z} \subset \mathbb{R}^n$. We say that $\phi(z) = (g(z), h(z))$ is a a *vector control barrier function* (VCBF) of $\mathcal{C}$ on $\mathcal{Z}$ relative to $\mathcal{U}$ if there exists $\alpha > 0$ such that the map $K_{\text{cbf}} : \mathbb{R}^n \rightrightarrows \mathcal{U}$, where

$$K_{\text{cbf}}(z) = \Big\{ u \in \mathcal{U} \mid \mathcal{L}_F g_i(z) + \sum_{\ell=1}^{r} u_\ell \mathcal{L}_{F_\ell} g_i(z) + \alpha g_i(z) \leq 0,$$
$$\mathcal{L}_F h_j(z) + \sum_{\ell=1}^{r} u_\ell \mathcal{L}_{F_\ell} h_j(z) + \alpha h_j(z) = 0,$$
$$1 \leq i \leq m, 1 \leq j \leq k \Big\},$$

takes nonempty values for all $z \in \mathcal{Z}$. In the special case where $m = 1$ and $k = 0$, the definition above coincides with the usual notion of a control barrier function [14] with a linear class $\mathcal{K}$ function.

If ϕ is a VCBF, and u is a feedback where $u(z) \in K_{\text{cbf}}(z)$ for all $z \in \mathcal{C}$, it follows that along solutions to (4), $\frac{d}{dt} g(z(t)) \leq -\alpha g(z(t))$ and $\frac{d}{dt} h(z(t)) = -\alpha h(z(t))$, which implies safety of $\mathcal{C}$. This is stated formally in the next result, which is generalization of [14, Corollary 2]. The result is stronger than Lemma 4.2, since it not only guarantees forward invariance of $\mathcal{C}$, but also that any trajectory starting at a point $\mathcal{Z} \setminus \mathcal{C}$, converges to $\mathcal{C}$ asymptotically.

Lemma 4.3: (VCBF-based Safeguarding Feedback): Consider the system (4) with safety set $\mathcal{C}$ defined in (2), and let $\phi = (g, h)$ be a vector control barrier function for $\mathcal{C}$ on $\mathcal{Z}$ relative to $\mathcal{U}$. Then the feedback controller $u : \mathcal{Z} \to \mathcal{U}$ is safeguarding if $u(z) \in K_{\text{cbf}}(z)$ for all $z \in \mathcal{Z}$, and the closed-loop system $\dot{z} = \mathcal{F}(z, u(z))$ admits unique solutions for all initial conditions.

Note that Lemma 4.3 slightly generalizes [14, Corollary 2] since it does not require Lipschitzness of the feedback controller (in fact, the uniqueness of solutions is enough to ensure safety via Nagumo's theorem).

To synthesize the safe-guarding feedback controller, one can pursue a design using a similar approach to Section IV-A. Given a cost function $J : \mathcal{Z} \times \mathcal{U} \to \mathbb{R}$, we let $u(z)$ solve the following mathematical program:

$$u(z) \in \underset{u \in K_{\text{cbf}}(z)}{\operatorname{argmin}} \Big\{ J(z,u) \Big\}. \tag{6}$$

Similar to the case of projection-based safeguarding feedback control, verifying the existence and uniqueness of solutions to the closed-loop system, as well as handling the situation where (6) does not have unique solutions, is nontrivial in general. These challenges can be addressed in the specific applications we consider in the following sections.

V. Design of Continuous-time Flows Solving Variational Inequalities

In this section, we use the control strategies of Section IV to design an algorithm, in the form of a continuous-time dynamical system, solving the variational inequality $\text{VI}(F, \mathcal{C})$, where $\mathcal{C}$ is given as (2), F is continuously differentiable, and g is twice continuously differentiable. Our approach views $\mathcal{C}$ as a safety set for the control-affine system

$$\begin{aligned} \dot{z} &= \mathcal{F}(z,u,v) \\ &= -F(z) - \sum_{i=1}^{m} u_i \nabla g_i(z) - \sum_{j=1}^{k} v_j \nabla h_j(z). \end{aligned} \tag{7}$$

Our design proceeds by synthesizing a feedback controller (u, v) to guarantee safety of $\mathcal{C}$ while ensuring convergence to $\text{SOL}(F, \mathcal{C})$. Before proceeding, we briefly explain the interpretation of the system (7). Based on the observation that $\dot{z} = -F(z)$ will find solutions to the unconstrained variational inequality $\text{VI}(F, \mathbb{R}^n)$, we set the drift term of (7) to $-F(z)$. However, flowing along this term might eventually violate the constraints. The inputs modify the flow of the drift to account for the constraints in a way that ensures that the solutions to (7) stay inside of or approach $\mathcal{C}$.

A. Projected Monotone Flow

We begin with a controller design following Section IV-A. Assuming MFCQ holds at $z \in \mathbb{R}^n$, the admissible inputs are

$$\begin{aligned} K_{\text{proj}}(z) = \Big\{ &(u,v) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^k \mid \\ &-\frac{\partial g_I}{\partial z}\frac{\partial g}{\partial z}^\top u - \frac{\partial g_I}{\partial z}\frac{\partial h}{\partial z}^\top v \leq \frac{\partial g_I}{\partial z} F(z), \\ &-\frac{\partial h}{\partial z}\frac{\partial g}{\partial z}^\top u - \frac{\partial h}{\partial z}\frac{\partial h}{\partial z}^\top v = \frac{\partial h}{\partial z} F(z) \Big\}. \end{aligned}$$

The following result states that the set of admissible controls is nonempty. We omit its proof for space reasons, but note that it readily follows from Farka's Lemma [18].

Lemma 5.1: (Projection onto Tangent Cone is Feasible): If MFCQ holds everywhere on $\mathcal{C}$, then $K_{\text{proj}}(z) \neq \emptyset$ for all $z \in \mathcal{C}$.

We set the objective function to be

$$J(z,u) = \frac{1}{2} \Big\| \sum_{i=1}^{m} u_i \nabla g_i(z) + \sum_{j=1}^{k} v_j \nabla h_j(z) \Big\|^2. \tag{8}$$

This function measures the magnitude of the "modification" of the drift term in (7). Thus, the QP-based controller (5) has the interpretation, at each z, of finding the control input such that the closed-loop system dynamics are as close as possible to $-F(z)$, while still being in $T_{\mathcal{C}}(z)$.

Note that the program (5) may not necessarily have unique solutions. However, the closed-loop dynamics of (7) is well defined regardless of which solution to (5) is chosen. In fact, we show next that the closed-loop system is equivalent to the system obtained by projecting $-F(z)$ onto $T_{\mathcal{C}}(z)$.

Proposition 5.2: (Equivalence of Closed-loop to Projected Dynamical System): Assume MFCQ holds at $z \in \mathcal{C}$ and let (u, v) be any solution to (5). Then, $\mathcal{F}(z, u, v) = \Pi_{T_{\mathcal{C}}(z)}(-F(z))$

The control-theoretic design outlined here results in the projected dynamical system $\dot{z} = \Pi_{T_{\mathcal{C}}(z)}(-F(z))$, which we refer to as the *projected monotone flow*. This system admits many equivalent descriptions, for example in terms of monotone differential inclusions, or complementarity systems [10], [11], [19], and its properties have been extensively studied [9]. The value of Proposition 5.2 stems from showing that safety-critical control can be used to systematically design algorithms that solve variational inequalities. Though the control strategy pursued in this section results in a known flow, this sets up the basis for employing other design strategies from safety-critical control to yield novel methods, as we show later.

Regarding the properties of the projected monotone flow, since unique solutions exist for all initial conditions [19, Chapter 3.2, Theorem 1(i)], forward invariance of $\mathcal{C}$ follows from Lemma 4.2. The equilibria of the projected monotone flow are equivalent to solutions to $\text{VI}(F, \mathcal{C})$, and are Lyapunov stable. Finally, stability of a solution z^* can be certified using the Lyapunov function $V(z) = \frac{1}{2}\|z - z^*\|^2$, as a consequence of [19, Chapter 3.2, Theorem 1(ii)].

Theorem 5.3: (Stability Properties of Projected Monotone Flow): Let $\mathcal{C}$ convex, and suppose MFCQ everywhere on $\mathcal{C}$. The following stability results hold for the projected monotone flow:

(i) $\mathcal{C}$ is forward invariant;
(ii) z^* is an equilibrium of the projected monotone flow if and only if $z^* \in \text{SOL}(F, \mathcal{C})$;
(iii) If $z^* \in \text{SOL}(F, \mathcal{C})$ and F is monotone, then z^* is globally Lyapunov stable relative to $\mathcal{C}$;
(iv) If $z^* \in \text{SOL}(F, \mathcal{C})$ and F is μ-strongly monotone, then z^* is globally asymptotically stable relative to $\mathcal{C}$.

B. Safe Monotone Flow

In this section, we design an algorithm to solve $\text{VI}(F, \mathcal{C})$ using the strategy outlined in Section IV-B. Since $\mathcal{C}$ is given by (2), we use $\phi = (g, h)$ as a VCBF. Letting $\alpha > 0$ be a design parameter, the set admissible controls is

$$\begin{aligned} K_{\text{cbf}}(z) = \Big\{ &(u,v) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^k \Big| \\ &-\frac{\partial g}{\partial z}\frac{\partial g}{\partial z}^\top u - \frac{\partial g}{\partial z}\frac{\partial h}{\partial z}^\top v \leq \frac{\partial g}{\partial z} F(z) - \alpha g(z) \\ &-\frac{\partial h}{\partial z}\frac{\partial g}{\partial z}^\top u - \frac{\partial h}{\partial z}\frac{\partial h}{\partial z}^\top v = \frac{\partial h}{\partial z} F(z) - \alpha h(z) \Big\}. \end{aligned}$$

One can show, cf. [20, Lemma 4.1], that if MFCQ holds everywhere on $\mathcal{C}$, then $K_{\text{cbf}}(z) \neq \emptyset$ for all z in an open set $\mathcal{Z}$

containing $\mathcal{C}$, which implies that ϕ is a valid VCBF for $\mathcal{C}$ on $\mathcal{Z}$ relative to $\mathcal{U} = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^k$.

We now design a controller for (7) using the optimization-based feedback (6), where the objective function J is given by (8). This controller has the same interpretation as before: determining the control input such that the closed-loop system dynamics are as close as possible to $-F(z)$. Similar to the case with projection-based methods, the problem (6) does not necessarily have unique solutions. Nonetheless, the closed-loop system is well-defined regardless of which solution is chosen. We refer to this system as the *safe monotone flow*. The following result summarizes its properties. The proof us similar to results in [20, Proposition 4.6, Proposition 5.2, Proposition 4.4].

Proposition 5.4: (Properties of Safe Monotone Flow): Assume MFCQ holds for everywhere on $\mathcal{C}$ and let $\mathcal{Z} \subset \mathbb{R}^n$ be an open set containing $\mathcal{C}$ on which K_{cbf} takes nonempty values. Then

- Let (u, v) solve (6) at $z \in \mathcal{Z}$. Then, $\mathcal{F}(z, u, v) = F_\alpha(z)$, where $F_\alpha(z) = \Pi_{T^{(\alpha)}_{\mathcal{C}}(z)}(-F(z))$ and

$$T^{(\alpha)}_{\mathcal{C}}(z) = \Big\{ \xi \in \mathbb{R}^n \mid \frac{\partial g(z)}{\partial z} \xi \leq -\alpha g(z), \frac{\partial h(z)}{\partial z} \xi = -\alpha h(z) \Big\}; \quad (9)$$

- F_α is locally Lipschitz continuous on $\mathcal{Z}$;
- For all $z \in \mathcal{C}$, $\lim_{\alpha \to \infty} F_\alpha(z) = \Pi_{T_{\mathcal{C}}(z)}(-F(z))$.

Proposition 5.4 highlights the relationship between the safe monotone flow and the projected monotone flow as well as their differences. In particular, the safe monotone flow can be interpreted as a projection of $-F(z)$ onto an approximation of the tangent cone at $\mathcal{C}$, $T^{(\alpha)}_{\mathcal{C}}(z)$. This approximation becomes exact as $\alpha \to \infty$. However, unlike the usual tangent cone, $T^{(\alpha)}_{\mathcal{C}}(z)$ is well defined for values outside of $\mathcal{C}$, and the projection $F_\alpha(z)$ is a Lipschitz map.

These differences mean that the safe monotone flow has two distinct advantages when compared with the projected monotone flow. First, unlike the projected monotone flow, the safe monotone flow is well-defined for infeasible initial conditions. Secondly, Lipschitz continuity of the dynamics ensures they can be numerically implemented using standard ODE discretization schemes.

Remark 5.5: The safe monotone flow is a generalization of the *safe gradient flow* introduced in [20]. While this system was originally studied in the context of nonconvex optimization, and local stability results were obtained, here we show that the same design methodology can be used to obtain algorithms solving monotone variational inequalities of the form $\text{VI}(F, \mathcal{C})$, and derive global stability results. •

We now move on to the problem of characterizing the equilibria of the closed-loop system under the safe monotone flow, and their stability properties. We show that the equilibria of the safe monotone flow are equivalent to solutions of $\text{VI}(F, \mathcal{C})$, and are stable. Though the proof has been omitted for space reasons, we remark that the stability properties can be verified using the following Lyapunov function:

$$V(z) = \frac{1}{2}\|z - z^*\|^2 - \frac{1}{\alpha^2} \inf_{\xi \in T^{(\alpha)}_{\mathcal{C}}(z)} \Big\{ \xi^\top F(z) + \frac{1}{2}\|\xi\|^2 \Big\}.$$

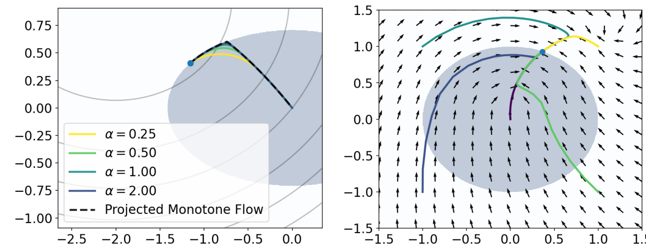

Fig. 1: (Left) Using the projected monotone flow and safe monotone flow to solve (10). Gray lines correspond to level sets of the objective function, and the shaded region is the feasible set. The solution to (10) is denoted by the blue dot. Colored trajectories correspond to solutions of the safe monotone flow with varying values of α, and the dashed trajectory is the solution to the projected monotone flow. (Right) Using the safe monotone flow to solve the game described in Section VI-B. We plot their vector field corresponding to the safe monotone flow with $\alpha = 0.1$. The shaded region is the feasible set. The Nash equilibrium is denoted by the blue dot. Colored trajectories correspond to solutions of the safe monotone flow from various initial conditions.

Theorem 5.6: (Stability Properties of Safe Monotone Flow): Suppose that $\mathcal{C}$ is convex, and MFCQ holds everywhere on $\mathcal{C}$. The following stability results hold for the safe monotone flow:

(i) $\mathcal{C}$ is forward invariant and asymptotically stable;
(ii) z^* is an equilibrium of the projected monotone flow if and only if $z^* \in \text{SOL}(F, \mathcal{C})$;
(iii) If $z^* \in \text{SOL}(F, \mathcal{C})$ and F is monotone, then z^* is globally Lyapunov stable relative to $\mathcal{C}$;
(iv) If $z^* \in \text{SOL}(F, \mathcal{C})$ and F is μ-strongly monotone, then z^* is globally asymptotically stable relative to $\mathcal{C}$.

VI. Numerical Examples

Here we illustrate our results on two example problems. The first is a simple convex optimization problem and the second is the problem of finding the Nash equilibria of a continuous game. We show that both problems can be expressed as monotone variational inequalities, and can be solved using the safe monotone flow.

A. Convex Optimization

Consider a simple quadratically constrained quadratic program of the form

$$\begin{aligned} \underset{z \in \mathbb{R}^n}{\text{minimize}} \quad & \frac{1}{2} z^\top Q_0 z + p_0^\top z \\ \text{subject to} \quad & \frac{1}{2} z^\top Q_i z + p_i^\top z \leq b_i \ \ \forall i \in \{1, \ldots, m\}. \end{aligned} \quad (10)$$

where $Q_0 \succ 0$ and $Q_i \succ 0$ for all $i = 1, \ldots, m$. In this case, (10) is a strongly convex optimization problem, and the solutions to the optimization problem are given by the set $\text{SOL}(F, \mathcal{C})$, where $F(z) = Q_0 z + p_0$ and

$$\mathcal{C} = \left\{ z \in \mathbb{R}^n \mid \frac{1}{2} z^\top Q_i z + p_i^\top z \leq b_i, i = 1, \ldots, m \right\}.$$

We search for the solutions using the projected monotone flow, and the safe monotone flow for various values of α. We illustrate our results in Figure 1(left). Since for this problem, F is μ-strongly monotone with $\mu = \lambda_{\min}(Q_0)$, the solution to (10) is unique. Furthermore by Theorem 5.3 and Theorem 5.6, the unique solution is globally asymptotically stable relative to $\mathcal{C}$ for both the projected monotone flow and the safe monotone flow corresponding to $\mathrm{VI}(F, \mathcal{C})$.

B. Nash Equilibrium Seeking

Consider a game with two players $i \in \{1, 2\}$, where each player seeks to minimize a cost function $J_i(z_1, z_2)$, subject to the constraint $z \in \mathcal{C}_i$. A point $z^* \in \mathcal{C}_1 \cap \mathcal{C}_2$ is called a *Nash equilibrium* [21] if

$$\begin{aligned} J_1(z_1^*, z_2^*) &\leq J_1(z_1, z_2^*) \qquad \forall z_1 \text{ such that } (z_1, z_2^*) \in \mathcal{C}_1 \\ J_2(z_1^*, z_2^*) &\leq J_2(z_1^*, z_2) \qquad \forall z_2 \text{ such that } (z_1^*, z_2) \in \mathcal{C}_2. \end{aligned}$$

Informally, a Nash equilibrium is a point where neither player can further reduce their cost by only changing their decision. The Nash equilibria of a game of this form can be characterized by solutions to the variational inequality $\mathrm{VI}(F, \mathcal{C})$, where F is the *pseudogradient operator*, $F(z) = (\nabla_{z_1} J_1(z), \nabla_{z_2} J_2(z))$, and $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$.

Here we consider a game where for $i \in \{1, 2\}$,

$$J_i(z_1, z_2) = \frac{1}{2}(z - \bar{z}^{(i)})^\top Q_i (z - \bar{z}^{(i)}),$$

with $\bar{z}^{(i)}$ chosen at random and

$$Q_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and $\mathcal{C}$ is the unit ball in $\mathbb{R}^2$. In this case, the pseudogradient operator is strongly monotone, so by Theorem 5.6 the unique Nash equilibrium is globally asymptotically stable with respect to the safe monotone flow. In Figure 1(right) show the vector field corresponding to the safe monotone flow with $\alpha = 0.1$, and plot several representative trajectories. Although Theorem 5.6 only guarantees that trajectories starting within $\mathcal{C}$ converge to the solution, those starting from infeasible initial conditions converge to the solution as well.

VII. Conclusions

We have shown how designing anytime algorithms to solve variational inequalities can be understood as a feedback control problem. Using techniques from safety-critical control, we have synthesized two continuous-time dynamics which find solutions to monotone variational inequalities. The first system is equivalent to projected methods, whereas the second system, termed safe monotone flow, is novel. For the latter, we have established stability of the equilibria (corresponding to the solutions of the variational inequality) and asymptotic stability in the case of strong monotonicity. Future work will study convergence of the safe monotone flow starting from infeasible conditions, establish conditions for convergence without the assumption of strong monotonicity, and develop methods for distributed problems. Finally, we hope to apply our methods to feedback optimization problems arising in applications such as power systems, traffic networks, communications systems.

References

[1] A. Hauswirth, S. Bolognani, G. Hug, and F. Dörfler, "Optimization algorithms as robust feedback controllers," *preprint*, 2021.

[2] L. S. P. Lawrence, J. W. Simpson-Porco, and E. Mallada, "Linear-convex optimal steady-state control," *IEEE Transactions on Automatic Control*, vol. 66, no. 11, pp. 5377–5384, 2021.

[3] S. H. Low, F. Paganini, and J. C. Doyle, "Internet congestion control," *IEEE Control Systems*, vol. 22, no. 1, pp. 28–43, 2002.

[4] G. Bianchin, J. Cortés, J. I. Poveda, and E. Dall'Anese, "Time-varying optimization of LTI systems via projected primal-dual gradient flows," *IEEE Transactions on Control of Network Systems*, vol. 9, no. 1, pp. 474–486, 2022.

[5] K. Arrow, L. Hurwitz, and H. Uzawa, *Studies in Linear and Non-Linear Programming*. Stanford, CA: Stanford University Press, 1958.

[6] R. W. Brockett, "Dynamical systems that sort lists, diagonalize matrices, and solve linear programming problems," *Linear Algebra and its Applications*, vol. 146, pp. 79–91, 1991.

[7] U. Helmke and J. B. Moore, *Optimization and Dynamical Systems*. Springer, 1994.

[8] H. Brezis, *Operateurs Maximaux Monotones et Semi-groupes de Contractions dans Les Espaces de Hilbert*. Elsevier, 1973.

[9] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications*, vol. 2 of *International Series in Operations Research and Management Science*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1996.

[10] B. Brogliato, A. Daniilidis, C. Lemaréchal, and V. Acary, "On the equivalence between complementarity systems, projected systems, and differential inclusions," *Systems & Control Letters*, vol. 55, no. 1, pp. 45–51, 2006.

[11] W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland, "Projected dynamical systems in a complementarity formalism," *Operations Research Letters*, vol. 27, pp. 83–91, 2000.

[12] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[13] P. Wieland and F. Allgöwer, "Constructive safety using control barrier functions," *IFAC Proceedings Volumes*, vol. 40, no. 12, pp. 462–467, 2007.

[14] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[15] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: theory and applications," in *European Control Conference*, (Naples, Italy), pp. 3420–3431, June 2019.

[16] W. Haddad and V. S. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton, NJ: Princeton University Press, 2008.

[17] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer, 2003.

[18] R. T. Rockafellar, *Convex Analysis*. Princeton University Press, 1970.

[19] J. P. Aubin and A. Cellina, *Differential Inclusions*, vol. 264 of *Grundlehren der mathematischen Wissenschaften*. New York: Springer, 1984.

[20] A. Allibhoy and J. Cortés, "Control barrier function based design of gradient flows for constrained nonlinear programming," `http://arxiv.org/abs/2204.01930`, 2022.

[21] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*. SIAM, 2 ed., 1999.